

On classes of graphs with logarithmic boolean-width

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Abstract

Boolean-width is a recently introduced graph parameter. Many problems are fixed parameter tractable when parametrized by boolean-width, for instance "Minimum Weighted Dominating Set" (MWDS) problem can be solved in $O^*(2^{3k})$ time given a boolean-decomposition of width k , hence for all graph classes where a boolean-decomposition of width $O(\log n)$ can be found in polynomial time, MWDS can be solved in polynomial time. We study graph classes having boolean-width $O(\log n)$ and problems solvable in $O^*(2^{O(k)})$, combining these two results to design polynomial algorithms. We show that for trapezoid graphs, circular permutation graphs, convex graphs, Dilworth- k graphs, circular arc graphs and complements of k -degenerate graphs, boolean-decompositions of width $O(\log n)$ can be found in polynomial time. We also show that circular k -trapezoid graphs have boolean-width $O(\log n)$, and find such a decomposition if a circular k -trapezoid intersection model is given. For many of the graph classes we also prove that they contain graphs of boolean-width $\Theta(\log n)$.

Further we apply the results from [1] to give a new polynomial time algorithm solving all vertex partitioning problems introduced by Proskurowski and Telle [23]. This extends previous results by Kratochvíl, Manuel and Miller [14] showing that a large subset of the vertex partitioning problems are polynomial solvable on interval graphs.

1 Introduction

One of the most studied problems in computer science is the classification of problems into complexity classes. For decades there has been done extensive work in order to decide which problems are solvable in polynomial time (in P). It is a common belief that NP-complete problems are not in P. One way to deal with problems we are unable to place in P is to classify on which inputs they can be solved in polynomial time. In particular we will in this paper study problems with simple, undirected graphs as input. To classify a problem as polynomial on a graph class one have to design an algorithm that given a graph, in polynomial time either confirms that this graph is not in the desired class or returns an optimal solution.

One way to generalize the concept of graph classes is by parametrizing the input. This means to partition all inputs into classes by assigning them a parameter. Tree-width, branch-width, clique-width and rank-width are some of the well known parameters for undirected graphs. One can then try to design an algorithm that runs in polynomial time for all graphs with parameter value below some limit k . If we find an algorithm with running time $n^{f(k)}$ we get polynomial running time for every fixed k . If we find an algorithm with running time $f(k) \times \text{poly}(n)$ we get polynomial running time even for some graphs with unbounded k , this is called an FPT-algorithm. In particular if we have an FPT algorithm with running time $2^{O(k)} \times \text{poly}(n)$ we will get polynomial algorithms when k is $O(\log n)$. We define a triple (graph class C , parameter W , problem P) to be polynomially parameter tractable (PPT) if:

1. Given an n -vertex graph G , we can in polynomial time compute a decomposition of G having W -width $f(n)$, or conclude that G is not in the class C .
2. Given a graph G and a decomposition of G having W -width $f(n)$ we can solve problem P on G in polynomial time.

For instance we show that the triple (convex graphs, boolean-width, MWDS) is PPT. Note that when both 1) and 2) both can be satisfied with same function f we can conclude that problem P is polynomial on C . However there are not many well known examples of such PPT triples satisfying both 1) and 2). We present many such triplets.

Boolean-width is a parameter recently introduced by Bui-Xuan, Telle and Vatshelle [4]. In this paper we study the class of graphs with boolean-width $O(\log n)$. We show that a large class of graphs including interval graphs, permutation graphs, convex graphs, circular k -trapezoid graphs, Dilworth k graphs and complement of planar graphs have boolean-width $O(\log n)$. Finally we show how to construct graphs by combining any of these types into bigger graphs all having boolean-width $O(\log n)$. For most of these classes we are able to show that the logarithmic bound is tight up to a constant factor, in particular we show that they contain graphs having rank-width $\Omega(\sqrt{n})$ and hence boolean-width at least $\frac{\log n}{2} - O(1)$. We do not have any polynomial recognition algorithm for graphs of boolean-width $O(\log n)$ in general. All of our proofs are constructive, but normally depend on having a certain representation of the graph as input. For many of the graph classes discussed in this paper the required representation can be found in polynomial time meaning we can in polynomial time build a decomposition given a graph belonging to the graph class, however some of the required representations are not known to be polynomially computable.

Combined with the results in [4] this leads to a polynomial algorithm for weighted dominating set for all the above mentioned graph classes (see Figure 1 for an overview). This unifies algorithms for minimum weighted dominating set on many graph classes. In fact we do not know any graph class where weighted dominating set is polynomial and the boolean-width is not $O(\log n)$. A constant approximation algorithm for finding a boolean-decomposition of width $O(\log n)$, would also unify the step of finding boolean decompositions and hence giving polynomial algorithms also for the classes where representations can not be found.

The (σ, ρ) and vertex partitioning problems which are covered by the framework introduced by Proskurowski and Telle [23] include among other Independent Set, Dominating Set, Perfect Code, k -Colouring, H -Cover and H -Homomorphism. Bui-Xuan et al.[1] showed that all these problems can be solved in $2^{O(k^2)} \times \text{poly}(n)$ given a boolean-decomposition of width k . Combining this with the results in this paper we get an $O(n^{O(\log n)})$ algorithm. In order solve these vertex partitioning problems in polynomial time for the graph classes we have discussed in this paper we must refine the running time analysis. In particular one needs to bound the number of d -neighbourhoods. This is done by bounding the size of a minimal representative. In all the graph classes we have studied in this paper we were able to build decompositions such that the minimal representatives have constant size. Hence all vertex partitioning problems are solvable in polynomial time on Dilworth k graphs, convex graphs, trapezoid graphs, circular permutation graphs, circular arc graphs and complement of k -degenerate graphs, and also when given a k -trapezoid model or a circular k -trapezoid model.

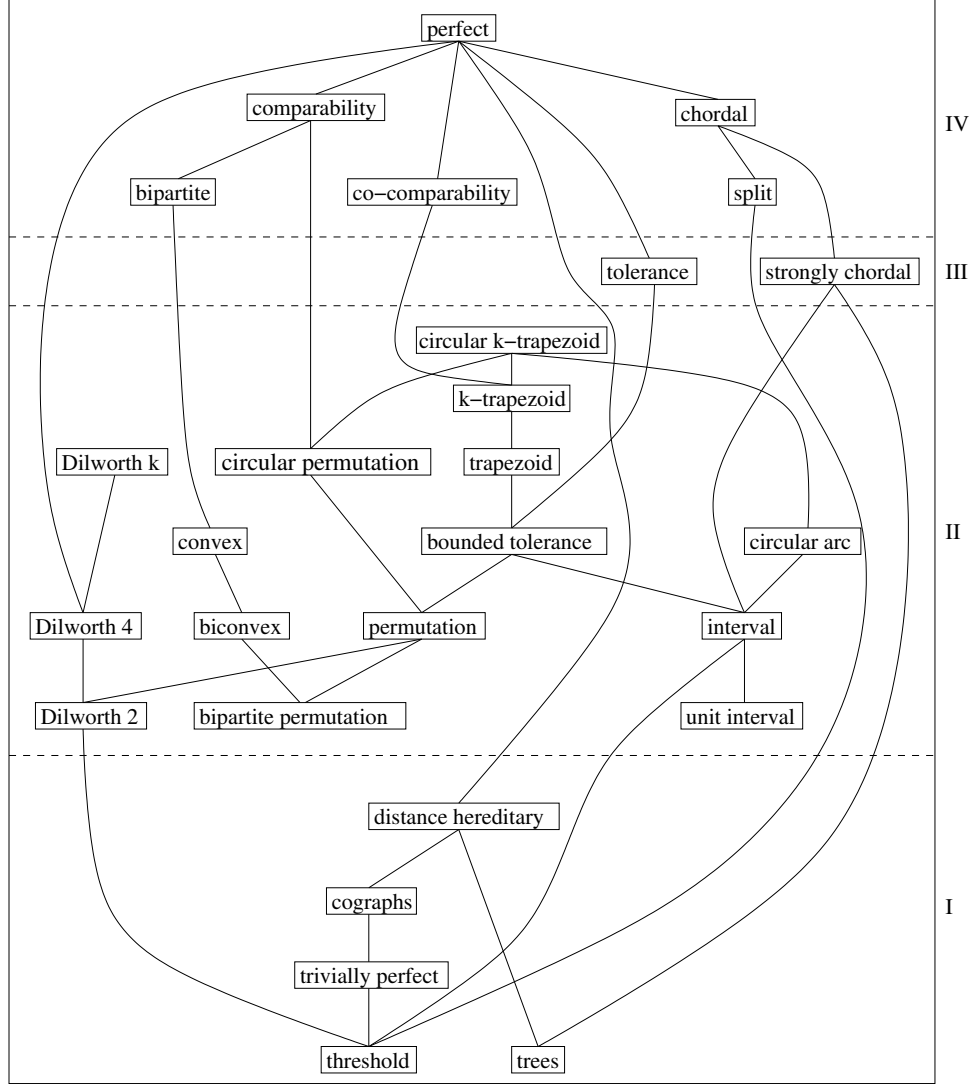


Figure 1: Inclusion diagram of some well-known graph classes. A link between a higher class A and a lower class B means that B is a subclass of A. (I) Graph classes where boolean-width is bounded by a constant. (II) Graph classes having boolean-width $O(\log n)$. (III) Boolean-width still unknown. (IV) There does not exist a boolean-decomposition of value $O(\log n)$, or it is NP-complete to compute it.

2 Framework

When applying divide-and-conquer to a graph we first need to divide the graph. A common way to store this information is to use a decomposition tree and to evaluate decomposition trees using a cut function. The following formalism is referred to as branch decomposition of a cut function and is standard in graph and matroid theory (see, e.g., [18, 10, 19]). Throughout the paper we will for $A \subseteq V(G)$ let \bar{A} denote the set $V(G) \setminus A$.

Definition 2.1 A decomposition tree of a graph G is a pair (T, δ) where T is a tree having

internal nodes of degree three and $n = |V(G)|$ leaves, and δ is a bijection between the vertices of G and the leaves of T . Every edge of T defines a cut $\{A, \bar{A}\}$ of the graph, i.e. a partition of $V(G)$ in two parts, namely the two parts given, via δ , by the leaves of the two subtrees of T we get by removing the edge. Let $f : 2^V \rightarrow \mathbb{R}$ be a symmetric function, i.e. $f(A) = f(\bar{A})$ for all $A \subseteq V(G)$, also called a *cut function*. The f -width of (T, δ) is the maximum value of $f(A)$, taken over all cuts $\{A, \bar{A}\}$ of G given by an edge uv of T . The f -width of G is the minimum f -width over all decomposition trees of G .

Caterpillar decompositions are decompositions where the underlying tree is a path with one leaf added to every internal node of the path. Many of our proofs will construct caterpillar decompositions. To describe a caterpillar decomposition, we only give an ordering of the vertices. To construct the caterpillar decomposition from an ordering, follow the path of the caterpillar and map the vertices to the leafs attached to the path.

The cuts $\{A, \bar{A}\}$ given by edges of the decomposition tree are used in the divide step of a divide-and-conquer approach. We solve the problem recursively, following the edges of the tree T (after choosing a root) in a bottom-up fashion, on the graphs induced by vertices of one side and of the other side of the cuts. In the conquer step we must join solutions from the two sides, and this is usually the most costly and complicated operation. Normally the conquer step uses Dynamic programming. The question of what 'solutions' we should store to get an efficient conquer step is related to what type of problem we are solving.

We suggest that the following equivalence relation on subsets of A will be useful for solving problems like dominating set.

Definition 2.2 Let G be a graph and $A \subseteq V(G)$. Two vertex subsets $X \subseteq A$ and $X' \subseteq A$ are *neighbourhood equivalent* w.r.t. A , denoted by $X \equiv_A X'$, if $\bar{A} \cap N(X) = \bar{A} \cap N(X')$.

In order to bound the number of neighbourhood equivalence classes we define boolean-width as follows:

Definition 2.3 (Boolean-width) The *cut-bool* : $2^{V(G)} \rightarrow \mathbb{R}$ function of a graph G is

$$\text{cut-bool}(A) = \log_2 |\{S \subseteq \bar{A} : \exists X \subseteq A \wedge S = \bar{A} \cap \bigcup_{x \in X} N(x)\}|$$

Using Definition 2.1 with $f = \text{cut-bool}$ we define the *boolean-width* of a decomposition tree, denoted $\text{boolw}(T, \delta)$, and the *boolean-width* of a graph, denoted $\text{boolw}(G)$.

It is known from Boolean matrix theory that *cut-bool* is symmetric [12].

Note that we take the logarithm base 2 of the number of equivalence classes simply to ensure that $0 \leq \text{boolw}(G) \leq |V(G)|$, which will ease the comparison of boolean-width to other parameters.

Rank-width was introduced by Oum and Seymour in [17, 18]. One way of defining rank-width is to replace the union in the definition of boolean-width by symmetric difference. The symmetric difference operator Δ applies to a family of sets and returns the set of elements appearing in an odd number of sets.

Definition 2.4 (Rank-width) The *cut-rank* : $2^{V(G)} \rightarrow \mathbb{N}$ function of a graph G is

$$\text{cut-rank}(A) = \log_2 |\{S \subseteq \bar{A} : \exists X \subseteq A \wedge S = \bar{A} \cap \bigtriangleup_{x \in X} N(x)\}| = \text{rk}(M_{A, \bar{A}})$$

Using Definition 2.1 with $f = \text{cut-rank}$ we define the rank-width of a decomposition tree, denoted $\text{rw}(T, \delta)$, and the rank-width of a graph, denoted $\text{rw}(G)$.

Now we establish some terminology used in this paper.

Definition 2.5 Let $G = (V, E)$ be a graph, (A, \overline{A}) a cut of G . Let $S = A \cap N(\overline{A})$. We define the middle vertices of a cut $m(A)$ as the vertices found by the following procedure: Let $S' = S$. While there are vertices $u, v \in S'$ such that $N(u) \setminus A = N(v) \setminus A$ remove u from S' . Return S' .

Note that $|m(A)| \leq n - 1$.

Many of the graphs we study in this paper are defined via intersection models.

Definition 2.6 An intersection model of a graph is a one to one mapping of objects to the vertices of a graph such that there is an edge between two vertices in the graph if and only if the objects mapped to the vertices intersect. Geometrical objects and sets are the most common examples of such objects.

3 Upper Bounds on Boolean-width of Graph Classes

3.1 Permutation graphs

Definition 3.1 A graph is a permutation graph if and only if it has an intersection model consisting of straight lines (one per vertex) between two parallels.

For more information on permutation graphs see [8].

Theorem 3.2 The boolean-width of a permutation graph G is at most $\log n$.

Proof We build a caterpillar decomposition by sorting the vertices by the upper endpoint of their corresponding line. Let us now consider a cut (A, \overline{A}) of the decomposition. Let σ be the total ordering of the vertices of $m(A)$ sorted by their lower endpoint, hence $\forall u, v \in m(A), \sigma(u) \leq \sigma(v)$ iff the lower endpoint of u is to the left of the lower endpoint of v . Since all upper endpoints of lines corresponding to vertices of A are to the left of all upper endpoints of lines corresponding to vertices of \overline{A} , two vertices $u \in A, u' \in \overline{A}$ are neighbours iff the lower endpoint of u is to the right of the lower endpoint of u' . Hence for any set $S \subseteq A$ there exists $x \in S$ such that $N(S) \cap \overline{A} = N(x) \cap \overline{A}$, namely the vertex of S with the rightmost lower endpoint. Then there are at most $m(A)$ neighbourhoods, and the theorem holds. \square

Lemma 3.3 [15] Given a graph, one can in linear time either decide that the graph is not a permutation graph, or output a permutation model for the graph.

3.2 Circular permutation graphs

Definition 3.4 A graph is a circular permutation graph if it has an intersection model consisting of curves between two distinct concentric circles, such that no two curves cross in more than one point, and no two curves touch without crossing.

For more information on circular permutation graphs see [20].

Theorem 3.5 *The boolean-width of a circular permutation graph G is at most $2 \log n$.*

A bound of $4 \log n$ follows from Theorem 3.11. The proof of the improved bound is similar to Theorem 3.2.

Lemma 3.6 [22] *Given a graph, one can in polynomial time either decide that the graph is not a circular permutation graph, or output a circular permutation model for the graph.*

3.3 k-Trapezoid graphs

Definition 3.7 *A graph is a k -trapezoid graph if it is the intersection graph of k -trapezoids, where a k -trapezoid is given by k intervals on k parallel lines.*

For more information on k -trapezoid graphs see [9].

Theorem 3.8 *The boolean-width of a k -trapezoid graph G is at most $k \log n$.*

The proof is similar to Theorem 3.11 and therefore was moved to the appendix.

It is NP-complete to compute a k -trapezoid model for a k -trapezoid graph for $k \geq 3$ (Yannakakis [26] and Flotow [9]).

Interval graphs are exactly the 1-trapezoid graphs.

Corollary 3.9 *The boolean-width of an interval graph G is at most $\log C$, where C is the size of the biggest clique in G .*

This holds since any set of intervals that contain a common point forms a clique. A proof can be found in appendix.

3.4 Circular k -trapezoid graphs

Definition 3.10 *A graph is a circular k -trapezoid graph if it is the intersection graph of circular k -trapezoids, where a circular k -trapezoid is given by k intervals on k concentric circles.*

For more information on circular k -trapezoid graphs see [21].

Theorem 3.11 *The boolean-width of a circular k -trapezoid graphs graph G is at most $2k \log n$.*

Proof We build a caterpillar decomposition by starting on a point p of the circle and by adding vertices of which k -trapezoid contains the point p of the innermost circle. We then order the k -trapezoids not containing p by the distance from p to the point of the k -trapezoid closest to p , and add the vertices in that order. Now, we look at a cut (A, \bar{A}) of the decomposition. We show that the boolean-width of any circular k -trapezoid graph is bounded by showing that any neighbourhood has a representative of size at most $2k$. Let us take any $S \subseteq A$ with $|S| > 2k$, we build a set $S' \subseteq S$ with $N(S) \cap \bar{A} = N(S') \cap \bar{A}$ and $|S'| \leq 2k$: For each line i , we take the 2 k -trapezoids r_i and l_i such that r_i is the k -trapezoid containing the point on the i^{th} line furthest in clockwise direction and similarly we choose l_i in counter-clockwise direction. We set S' as the set of all r_i and l_i . Let us assume for contradiction that $\exists x \in \bar{A} : x \in N(S) \setminus N(S')$. There must exist some j such that the k -trapezoid of x intersects some trapezoid of S on the j^{th} line. Either it intersects r_j or l_j , else the whole trapezoid of x is contained in the area between r_j and l_j , then by construction of the decomposition, x would have been in A . Thus $N(S) \cap \bar{A} = N(S') \cap \bar{A}$, and hence any neighbourhood has a representative of size at most $2k$ and the theorem holds. \square

Circular arc graphs are exactly circular 1-trapezoid graphs.

Corollary 3.12 *The boolean-width of a circular arc graph G is at most $2\log C + 2$, where C is the size of the biggest clique in G .*

This holds since any set of arcs that contain a common point forms a clique. A proof can be found in appendix.

3.5 Convex graphs

Definition 3.13 *An ordering $<$ of X in a bipartite graph $B = (X, Y, E)$ has the adjacency property if for every vertex y in Y , $N(y)$ consists of vertices that are consecutive (an interval) in the ordering $<$ of X . A bipartite graph (X, Y, E) is convex if there is an ordering of X or Y that satisfies the adjacency property.*

For more information about convex graphs see [24].

Theorem 3.14 *The boolean-width of a convex graph G is at most $\log n$.*

Proof Since G is convex we know there exists a bipartition (X, Y) of V and σ_X an ordering of X such that $\forall u \in Y, x, y \in N(u)$ we have $\forall z \in X : \sigma_X(x) < \sigma_X(z) < \sigma_X(y)$ then $z \in N(u)$. We construct a total ordering σ of V from σ_X by keeping the ordering of vertices in X and for each vertex $v \in Y$ we insert v immediately after the last element of $N(v)$. We construct a caterpillar decomposition based on the order σ . Consider a cut (A, \bar{A}) in the caterpillar decomposition. Then $m(A) \cap Y = \emptyset$ by construction of σ .

Let v_1, v_2, \dots, v_t be the ordering of the vertices of $m(A)$ induced by σ , ($t = |m(A)|$). Since all the vertices in $Y \cap \bar{A}$ appear later in σ than v_t , they all either see v_t or have no neighbours in A . By the property of a convex graph if some vertex $u \in Y \cap \bar{A}$ see v_i then u also see v_{i+1} . Hence we get $\text{boolw}(G) \leq \log(|m(A)| + 1)$ and since $|m(A)| \leq n - 1$ the theorem holds. \square

Lemma 3.15 [3] *Given a graph, one can in polynomial time either decide that the graph is not a convex graph, or output an ordering verifying that the graph is convex.*

3.6 Graphs of bounded Dilworth number

Definition 3.16 (Dilworth number) *Two vertices x and y are said to be comparable if either $N(y) \subseteq N[x]$ or $N(x) \subseteq N[y]$. The Dilworth number of a graph is the largest number of pairwise incomparable vertices of the graph. A graph is a Dilworth k graph if it has Dilworth number k .*

In order to prove our next result, we will need the following theorem, well known as Dilworth's theorem from posets theory:

Theorem 3.17 (Dilworth [7]) *In a finite partial order, the size of a maximum anti-chain is equal to the minimum number of chains needed to cover its elements.*

Theorem 3.18 *The boolean-width of a Dilworth k graph G is at most $k \log n$.*

Proof From Dilworth's theorem, we know that if a graph G is a Dilworth k , then the Hasse diagram associated with the inclusion of neighbourhoods of G can be covered with k chains. We call these chains C_1, \dots, C_k . Thus, we build a caterpillar decomposition by adding the vertices of C_1 , then the vertices of C_2 , and so on. Let us now take any cut (A, \bar{A}) of the decomposition. $\forall S \subseteq A$, we call X_S the subset of S such that $\forall x \in S, \exists y \in X_S : N(x) \subseteq N(y)$. Thus we have $N(S) = N(X_S)$. Moreover, $|X_S| \leq k$ (at most one vertex for each chain). Therefore there can be at most n^k neighbourhoods and the result holds. \square

3.7 Complements of k -degenerate graphs

Definition 3.19 A graph G is k -degenerate if and only if there exists an elimination ordering on the vertices such that every vertex has at most k neighbours appearing later in the ordering.

Theorem 3.20 The boolean-width of a graph G , where \bar{G} is a k -degenerate graph, is at most $k \log n$.

Proof We build a caterpillar decomposition of G using the elimination ordering induced by the k -degeneracy of \bar{G} . We consider a cut (A, \bar{A}) of the decomposition. Since \bar{G} is k -degenerate, we have $\forall x \in A, d(x) \leq k$ in \bar{G} , therefore $\forall x \in A, |N(x) \cap \bar{A}| \geq |\bar{A}| - k$ in G , and thus $|\{N(x) \cap \bar{A} : x \in A\}| \leq \binom{n}{k}$. Hence, we get the result $boolw(G) \leq \log \binom{n}{k} \leq k \log n$. \square

4 Lower Bounds on Boolean-width of Graph Classes

In the previous section we showed many classes of graphs with boolean-width $O(\log n)$, now we want to show that a big class of graphs have low boolean-width and high rank-width.

A Hsu-graph is a bipartite graph $H = (V, E)$ where $V = \{v_1, v_2, \dots, v_a\}, \{u_1, u_2, \dots, u_b\}$ where $v_i, u_j \in E(H) \Leftrightarrow i \leq j$.

A Hsu-join-chain of length q and width p is constructed as follows: Let $\mathcal{F} = G_1, G_2, \dots, G_q$ be a family of connected graphs, all on at least p vertices and boolean-width $O(\log p)$. For each graph G_i pick a set S_i of p vertices. For each pair G_i, G_{i+1} connect S_i to S_{i+1} by a Hsu-graph.

Theorem 4.1 Let G be a Hsu-join-chain of length q and width p where $q > 3p$ then $boolw(G) \in O(\log p)$ and $rank-width(G) \geq p/2$.

Proof Let $\mathcal{F} = G_1, G_2, \dots, G_q$ be the family of graphs used to construct G . For each graph G_i take an optimal rooted decomposition tree and identify the roots with leaves of a caterpillar using the order G_1, G_2, \dots, G_q . For a cut (A, \bar{A}) where $A \subseteq V(G_i)$ for some i we know there are $2^{O(\log p)}$ neighbourhoods in the cut $(A, V(G_i) \setminus A)$, $O(p)$ neighbourhoods in the cut $(A, V(G_{i-1}))$ and $O(p)$ neighbourhoods in the cut $(A, V(G_{i+1}))$. This means that boolean-cut value of (A, \bar{A}) is $O(\log p)$. For the cuts $(V(G_1) \cup V(G_2) \cup \dots \cup V(G_i), V(G_{i+1}) \cup \dots \cup V(G_q))$ only $V(G_i) \cup V(G_{i+1})$ have neighbours on each side and hence the cut edges form a Hsu-graph and hence the boolean-width of G is $O(\log p)$. To show that G has high rank-width we use the fact that there must be a $(\frac{1}{3}, \frac{2}{3})$ -balanced cut (A, \bar{A}) and we show that every such cut has rank-width at least $p/2$. Assume for contradiction that the rank-width of (A, \bar{A}) is less than $p/2$, then there are at most p graphs in \mathcal{F} intersecting both A and \bar{A} . Since otherwise in each G_i intersecting both sides of the cut we can find a crossing edge. Taking the edges from all G_i such that i is odd (resp. even)

will give an induced matching of size at least $p/2$ and hence the rank-width of the cut would be at least $p/2$. Since the cut is balanced we know that there is some graph in \mathcal{F} completely contained in A and some graph in \mathcal{F} completely contained in \bar{A} . Otherwise rank-width would be at least $q/6 > p/2$. For each cut (G_i, G_{i+1}) we know that there is an induced subgraph of the cut isomorphic to a Hsu-graph of size $|S_{i+1} \cap A| - |S_i \cap A|$. Combining all the Hsu-graphs obtained from either the cuts with i odd or i even ensures that the cut has rank-width at least $p/2$. \square

Corollary 4.2 *There exists an infinite family of bipartite permutation graphs with rank-width $\Omega(\sqrt{n})$.*

Let a Hsu-Stable-chain of length q and width p be the Hsu-join-chain of length q and width p where $\forall i, G_i$ is a stable set of size p . A Hsu-Stable-chain is a bipartite permutation graph with boolean-width $\Theta(n)$.

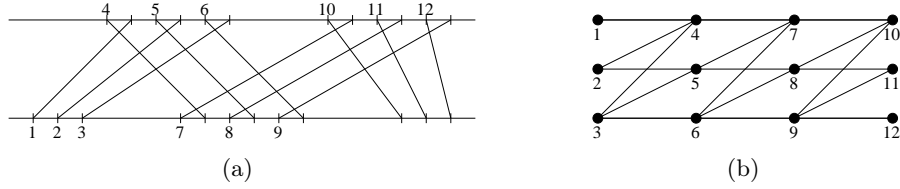


Figure 2: Hsu-Stable chain 3×4 and its permutation representation.

Corollary 4.3 *There exists an infinite family of unit interval graphs (i.e. an interval graph having all intervals of unit length) with rank-width $\Omega(\sqrt{n})$.*

Let a Hsu-Clique-chain of length q and width p be the Hsu-join-chain of length q and width p where $\forall i, G_i$ is a Clique of size p . A Hsu-Clique-chain is a unit interval graph with boolean-width $\Theta(n)$.

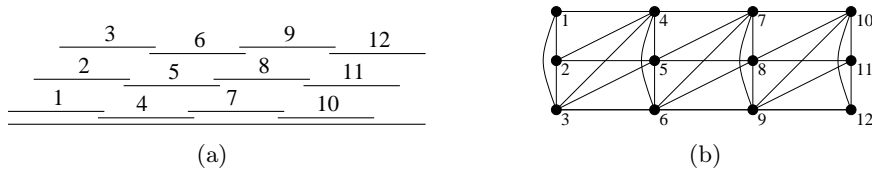


Figure 3: Hsu-Clique chain 3×4 and its unit interval representation.

Remark 4.4 *Let G be a graph which belongs to a class where there exists a problem solvable in time $O^*(2^{O(\text{boolw}(G))})$ which is NP-complete, then either G does not have a boolean decomposition of value $O(\log n)$, or it is NP-complete to build such a decomposition.*

It implies that, unless $P=NP$, one cannot compute a boolean decomposition of value $O(\log n)$ in polynomial time for any of the following graph classes:

- Split graphs (from [2])
- Circle graphs (from [11])

- Co-comparability graphs (from [5])
- Chordal bipartite graphs (from [16])

5 Vertex partitioning problems

In [23] Telle and Proskurowski introduced a generalized framework for handling many types of vertex subset and vertex partitioning problems in a unified manner. These types of problems have been studied in many ways. Normally they are described by a degree constraint matrix although there are also problems not describable by a degree constraint matrix.

Definition 5.1 *A degree constraint matrix D_q is a q by q matrix with entries being finite or co-finite subsets of natural numbers. A D_q -partition in a graph G is a partition $\{V_1, V_2, \dots, V_q\}$ of $V(G)$ such that for $1 \leq i, j \leq q$ we have $\forall v \in V_i : |N(v) \cap V_j| \in D_q[i, j]$.*

We call the vertex partitioning problems describable by a degree constraint matrix for D_q -problems. Telle and Proskurowski showed that all D_q -problems are solvable in FPT time parametrized by tree-width[23]. Kobler and Rotics showed that D_q -problems are solvable on graphs of bounded clique-width[13], and with a little effort their algorithm can be made into an FPT algorithm. Bui-Xuan et al showed that D_q -problems are FPT when parametrized by boolean-width[1]. Kratochvil et al. [14] showed that a large subset of the D_q -problems are solvable in polynomial time on interval graphs. We generalize the results of [14] by showing that all D_q -problems are solvable in polynomial time on many well known graph classes.

We will build on the algorithm of Bui-Xuan[1], there the bottleneck for running time is the number of equivalence classes of d -neighbourhoods.

Definition 5.2 (d -neighbour equivalence) Let G be a graph and $A \subseteq V(G)$ a vertex subset of G . Two vertex subsets $X \subseteq A$ and $X' \subseteq A$ are d -neighbour equivalent w.r.t. A , denoted by $X \equiv_A^d X'$, if

$$\forall v \in A : (|N(v) \cap X| = |N(v) \cap X'|) \vee (|N(v) \cap X| \geq d \wedge |N(v) \cap X'| \geq d).$$

The integer value d depends on the sets used in the degree constraint matrix. Let $d(\mathbb{N}) = 0$. For every finite or co-finite non-empty set $\mu \subseteq \mathbb{N}$, let $d(\mu) = 1 + \min(\max x : x \in \mu, \max x : x \notin \mu)$. For a matrix D_q , the value d will be $\max d(\mu) : \mu \in D_q$.

Let UN_d be the maximum number of equivalence classes of the d -neighbourhood equivalence relation over the cuts of a decomposition, the running time of the algorithm given in [1] will be $O(n(m + qd * \text{boolw}(G) * UN_d^3 * 2^{\text{boolw}(G)}))$. What we need in order to give a polynomial algorithm for all D_q -problems on a specific graph class is to give a decomposition with UN_d bounded by some polynomial in n for each cut and boolean-width by $O(\log n)$. One way to do this is to show that every d -neighbourhood has a representative of constant size in the $O(\log n)$ boolean-decomposition.

In all the proofs of this paper we bound the boolean-width by bounding the size of the representatives needed. More formally, for every $A \subseteq V(G)$ given by the decomposition tree, we showed that for every $S \subseteq A$ there exists a set $R \subseteq S$ such that $N(S) \cap \bar{A} = N(R) \cap \bar{A}$. In order to bound boolean-width we do not need $R \subseteq S$, but this will be crucial to bound the number of d -neighbourhoods.

Lemma 5.3 *Given $A \subseteq V(G)$, assume $\forall S \subseteq A, \exists R \subseteq S$ such that $N(S) \cap \overline{A} = N(R) \cap \overline{A}$ and there exists an integer k such that $|R| \leq k$. Then, $\exists R' \subseteq S$ such that $R' \equiv_A^d S$ and $|R'| \leq dk$.*

Proof Proof by induction on d . 1-neighbourhoods are exactly the same as neighbourhoods, hence the lemma is trivially true for $d = 1$. Assume the statement of the lemma is true for all values up to $d - 1$. Let S and R be sets satisfying the conditions in the lemma. Find minimal $X \subseteq S$ such that $S \setminus R \equiv_A^{d-1} X$, then by induction hypothesis, $|X| \leq (d - 1)k$. Let $R' = R \cup X$. Now it is easy to see that $R' \equiv_A^d S$. \square

Combining the above lemma with the results in Section 3, especially the fact that neighbourhoods can be described by a representative of constant size we get the following theorem.

Theorem 5.4 *All D_q -problems are solvable in polynomial time on the following graph classes: Dilworth k graphs, convex graphs, trapezoid graphs, circular permutation graphs, circular arc graphs and complement of k -degenerate graphs.*

Theorem 5.5 *Given a k -trapezoid model or a circular k -trapezoid model of G as input. All D_q -problems are solvable in polynomial time on G .*

6 Conclusion

We have shown that a large family of graph classes including interval graphs and trapezoid graphs have low boolean-width on one hand, while on the other hand, they have high rank-width and we can easily find a decomposition of low boolean-width. This means for instance that the best algorithms for the minimum dominating set problem using boolean-decompositions are polynomial while the best known using rank-decompositions are not. Moreover, since the size of the representative is bounded by a constant for all these graph classes, we can improve the analysis of the running-time of the algorithm for vertex partitioning problems in [1]. Thus, we have provided many graph classes for which every finite vertex partitioning problem can be solved in polynomial time, as well as their weighted version.

Question 1 *Is there any graph class having boolean-width $\Omega(\log n)$ where weighted dominating set is polynomially solvable?*

Question 2 *For tolerance graphs, both the complexity class of Dominating Set and the value of boolean-width are unknown.*

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A Appendix

Interval graphs

Definition A.1 *A graph is an interval graph if it has an intersection model consisting of intervals on a straight line.*

Proof of Corollary 3.9. **Proof** Any interval graph has an interval representation where no interval starts or ends at the same point. We build a caterpillar decomposition by sorting the vertices by the left endpoint of their corresponding intervals. Let us now consider a cut (A, \bar{A}) of the decomposition. Let σ be the total ordering of the vertices of $m(A)$ sorted by their right endpoint. Since all left endpoints of intervals corresponding to vertices of A are to the left of all left endpoints of intervals corresponding to vertices of \bar{A} , two vertices $u \in A, u' \in \bar{A}$ are neighbours iff the right endpoint of u is to the right of the left endpoint of u' . Hence $\forall u, v \in A$: if $\sigma(u) \leq \sigma(v)$ then $N(u) \cap \bar{A} \subseteq N(v) \cap \bar{A}$. Then $\beta w(A) \leq \log(|m(A)| + 1)$. Since $m(A)$ is a clique in G and there is one element in \bar{A} neighbouring all vertices in $m(A)$, we have $|m(A)| \leq C - 1$ hence the theorem holds. \square

Circular arc graphs

Definition A.2 *A circular arc graph is the intersection graph of arcs of a circle.*

For more information on circular arc graphs see [25].

Proof of Corollary 3.12 **Proof** We define the starting (resp. ending) point of an arc as the first (resp. last) point encountered when going around the circle in clockwise direction, starting at a fixed point p . We build a caterpillar decomposition by ordering the vertices according to the startingpoint of their corresponding arc. Now, we look at a cut (A, \bar{A}) of the decomposition. We show that the boolean-width of any circular arc graph is bounded by showing that for any $S \subseteq A$ there is a set $S' \subseteq S$ of size at most 2 such that $N(S) \cap \bar{A} = N(S') \cap \bar{A}$. Assume $|S| > 2$, let S' be the set containing the element of S having the first startingpoint and the element of S having the last endpoint when traversing clockwise starting from p . Since no vertex of \bar{A} can correspond to an arc properly contained in the section between p and the starting point of the element of A with the latest starting point, we have that $\forall u \in \bar{A}, u \in N(S)$ iff $u \in N(S')$. Since $m(A)$ is the union of two cliques in G and there exist two elements in \bar{A} neighbouring all vertices in $m(A)$, we have $|m(A)| \leq 2(C - 1)$. Thus, there can be at most $(2C)^2$ neighbourhoods and the theorem holds. \square

Circular permutation graphs

Proof of Theorem 3.5 **Proof** We build a caterpillar decomposition using σ_i , an ordering obtained by sorting the vertices by the inner endpoint of their corresponding line in clockwise order starting with any point. Let us now consider a cut (A, \bar{A}) of the decomposition. Let l be the unique line and v the corresponding vertex such that:

1. $v \in A$
2. All lines corresponding to vertices appearing before v in σ_i cross l in a clockwise direction.

3. No line corresponding to a vertex in A appearing after v in σ_i cross l in a counter-clockwise direction.

Let v be the first vertex of σ_o , an ordering of the vertices of $m(A)$. Continue the ordering by repeating the two steps above for the vertices in A not yet in σ_o .

Since all inner endpoints of lines corresponding to vertices of A are consecutive on the inner cycle, if a vertex $u \in \bar{A}$ is neighbour with a vertex $v \in A$ then either u is neighbour with all vertices before v in σ_o or u is neighbour with all vertices after v in σ_o .

If $S \subseteq A$ is a minimal set then $|S| \leq 2$. Assume for contradiction that there are at least 3 elements in S . Let x, y, z be three elements of S such that $\sigma_o(x) < \sigma_o(y) < \sigma_o(z)$. Now any vertex in $N(y) \cap \bar{A}$ will see either x or z . Hence $N(S \setminus y) = N(S)$ contradicting minimality of S .

The number of neighbourhoods is at most n^2 , hence the theorem holds. \square

Trapezoid graphs

Definition A.3 *A graph is a trapezoid graph if it is the intersection graph of trapezoids between two parallel lines.*

For more information on trapezoid graphs see [6].

Theorem A.4 *The boolean-width of a trapezoid graph G is at most $2 \log n$.*

Proof We build a caterpillar decomposition by sorting the vertices by the upper right corner of their corresponding trapezoid from left to right. Let us now consider a cut (A, \bar{A}) of the decomposition. We show that the boolean-width of any trapezoid graph is bounded by showing that any neighbourhood has a representative of size at most 2. Let us take any $S \subseteq A$ with $|S| > 2$, we build a set $S' \subseteq S$ with $N(S) \cap \bar{A} = N(S') \cap \bar{A}$ and $|S'| \leq 2$: for the upper line (resp. lower), we take the the trapezoid u (resp. l) with the rightmost upper (resp. lower) right corner, we then set $S' = \{u, l\}$. Let us assume for contradiction that $\exists x \in \bar{A} : x \in N(S) \setminus N(S')$. The trapezoid of x must intersect some trapezoid of S on the upper or lower line. If it does not intersect u or l , then the whole trapezoid of x is to the right of u and l . By construction of the decomposition, x would have been in A . Thus $N(S) \cap \bar{A} = N(S') \cap \bar{A}$, and hence any neighbourhood has a representative of size at most $k + 1$ and the theorem holds. \square

k-trapezoid graphs

Proof of Theorem 3.8 **Proof** We build a caterpillar decomposition by sorting the vertices by the rightmost corner of their corresponding k -trapezoid. Let us now consider a cut (A, \bar{A}) of the decomposition. We show that the boolean-width of any k -trapezoid graph is bounded by showing that any neighbourhood has a representative of size at most k . Let us take any $S \subseteq A$ with $|S| > k$, we build a set $S' \subseteq S$ with $N(S) \cap \bar{A} = N(S') \cap \bar{A}$ and $|S'| \leq k$: for each line i , we take the the k -trapezoid r_i such that r_i is the k -trapezoid corresponding to a vertex in S containing the rightmost point on the i^{th} line. We set S' as the set of all r_i and l_i . Let us assume for contradiction that $\exists x \in \bar{A} : x \in N(S) \setminus N(S')$. There must exist some j such that the k -trapezoid of x intersects some trapezoid of S on the j^{th} line. If it does not intersect r_j , then the whole trapezoid of x is to the right of r_j . By construction of the decomposition, x would have been in A . Thus $N(S) \cap \bar{A} = N(S') \cap \bar{A}$, and hence any neighbourhood has a representative of size at most k and the theorem holds. \square

Lower bounds

Proof of Corollary 4.2 **Proof** Let a Hsu-Stable-chain of length q and width p be the Hsu-join-chain of length q and width p where $\forall i, G_i$ is a stable set of size p . We now have to show that every Hsu-Stable-chain is a bipartite permutation graph. Since it is trivial that every Hsu-Stable-chain is a bipartite graph (we put the G_i in one colour class when i is even, and in the other colour class when i is odd), we just have to build the permutation representation of any Hsu-Stable-chain. For any $i \in \{1, \dots, q\}$, we represent G_i as a set of p parallel lines, thus building a stable set of size p , the sets of lines going alternatively top-down and bottom up, like shown on figure 2(a). We then make the lines cross by extending the scheme shown on figure 2(a): for any G_i the j^{th} line is crossed by the j last lines of the $(j-1)^{th}$ set and the j first lines of the $(j+1)^{th}$ set, hence building a Hsu-Stable-chain. \square

Proof of Corollary 4.3 **Proof** Let a Hsu-Clique-chain of length q and width p be the Hsu-join-chain of length q and width p where $\forall i, G_i$ is a K_p . We now have to show that every Hsu-Clique-chain is a unit interval graph, which we do by building the unit interval representation of a $p \times q$ Hsu-Clique-chain. We build each K_p of the Hsu-Clique-chain using p intervals, each one being slightly shifted to the right w.r.t. the previous one, like shown on figure 3(a). We then put each set of intervals just next to the previous one, without having the intervals at the same height overlapping: for any G_i the j^{th} line is crossed by the j last lines of the $(j-1)^{th}$ set and the j first lines of the $(j+1)^{th}$ set, hence building a Hsu-Clique-chain. \square